

## NOTES ON THE VECTOR ADELIC GRASSMANNIAN

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ABSTRACT. These are miscellaneous private notes about the generalization of [W2] to the vector case. Some parts date from around 1998 and are fairly reliable; others are in a very rough state, and may contain errors.

## 1. THE CALOGERO-MOSER SPACES

Fix two positive integers  $n$  and  $r$ . We start off with the complex vector space  $V \equiv V(n, r)$  of quadruples  $(X, Y; v, w)$  where  $X$  and  $Y$  are  $n \times n$  matrices,  $v$  is  $n \times r$  and  $w$  is  $r \times n$ . The trace form  $\text{tr} AB$  gives a nonsingular pairing between the spaces of  $p \times q$  and  $q \times p$  matrices; in this way we may consider  $V$  as the cotangent bundle of the space of pairs  $(X, v)$ , thinking of  $(Y, w)$  as a cotangent vector at  $(X, v)$ . Thus  $V$  is equipped with the (holomorphic) symplectic form

$$\text{tr}(dY \wedge dX + dw \wedge dv) .$$

The group  $GL(n, \mathbb{C})$  acts symplectically on  $V$  by

$$g(X, Y; v, w) := (gXg^{-1}, gYg^{-1}; gv, wg^{-1}) ;$$

the moment map  $\mu : V \rightarrow \mathfrak{gl}(n, \mathbb{C})$  for this action is

$$\mu(X, Y; v, w) = [X, Y] + vw .$$

The action of  $GL(n, \mathbb{C})$  on  $\mu^{-1}(-I)$  is free, so we can form the symplectic quotient

$$\mathcal{C} \equiv \mathcal{C}_n(r) := \mu^{-1}(-I)/GL(n, \mathbb{C}) .$$

For  $r = 1$  this is the space studied in [W2]. As in that case,  $\mathcal{C}$  is a smooth irreducible affine algebraic variety; in fact it is a hyperkähler variety (see [N]), but that will not be important here.

We denote by  $\mathcal{C}'$  the dense open subset of  $\mathcal{C}$  consisting of classes of quadruples  $(X, Y; v, w)$  where  $X$  has distinct eigenvalues. Fix an orbit representative with  $X$  diagonal, say  $X = \text{diag}(x_1, \dots, x_n)$ . Let  $\{v_1, \dots, v_n\}$  be the rows of  $v$ , and let  $\{w_1, \dots, w_n\}$  be the columns of  $w$ . Equating the diagonal entries in the equation  $[X, Y] + vw = -I$  gives

$$(1.1) \quad v_i w_i = -1 \quad \text{for } 1 \leq i \leq n ,$$

and equating the nondiagonal entries gives

$$(1.2) \quad Y_{ij} = - \frac{v_i w_j}{x_i - x_j} \quad \text{for } i \neq j .$$

So just as in the case  $r = 1$ , the diagonal entries  $\alpha_i$  of  $Y$  are free, and the off-diagonal part of  $Y$  is determined by  $(X, v, w)$ . For  $r = 1$ , we can use the action of the diagonal matrices in  $GL(n, \mathbb{C})$  to normalize all the (scalars)  $v_i = 1$  (and

$w_i = -1$ ). As we see from (1.1), the corresponding statement for  $r > 1$  is that each pair  $(v_i, w_i)$  determines a point of the hyperkähler variety

$$A_r := \{(\xi, \eta) : \xi\eta = -1\}/\mathbb{C}^\times,$$

where  $\xi$  and  $\eta$  are row and column vectors of length  $r$ , and  $\lambda \in \mathbb{C}^\times$  acts by  $(\xi, \eta) \mapsto (\lambda\xi, \lambda^{-1}\eta)$ .

Now, our orbit representative is unique up to the action of a diagonal matrix  $D := \text{diag}(d_1, \dots, d_n)$  and then of a permutation matrix. The action of  $D$  is given by

$$y_{ij} \mapsto d_i d_j^{-1} y_{ij}, \quad v_i \mapsto d_i v_i, \quad w_i \mapsto d_i^{-1} w_i$$

(the parameters  $(x_i)$  and  $(\alpha_i)$  staying fixed). This means that  $\mathcal{C}'$  is the quotient by the symmetric group  $\Sigma_n$  of the space

$$(\mathbb{C}^n \setminus \Delta) \times \mathbb{C}^n \times A_r^n$$

with parameters  $(x_i, \alpha_i, q_i)$ , where we have set  $q_i := (v_i, w_i) \bmod \mathbb{C}^\times$ . Further,  $\Sigma_n$  acts by simultaneous permutation of the parameters  $(x_i, \alpha_i, q_i)$ . Gibbons and Hermesen interpret this as saying that  $\mathcal{C}'$  is the phase space for a system of  $n$  indistinguishable particles (located at the points  $x_i$ ), each having the variety  $A_r$  as space of “internal degrees of freedom”.

*Remark.* More important for us will be the similar subspace  $\mathcal{C}^d$  of  $\mathcal{C}$ , where  $Y$  has distinct eigenvalues;  $\mathcal{C}'$  and  $\mathcal{C}^d$  are interchanged by the bispectral involution (7.2).

## 2. THE GIBBONS-HERMSEN HIERARCHY

On the symplectic space  $V$  of quadruples  $(X, Y; v, w)$  we consider the  $GL(n, \mathbb{C})$ -invariant Hamiltonians

$$(2.1) \quad J_{k, \alpha} := \text{tr} Y^k v \alpha w, \quad k \geq 0, \alpha \in \mathfrak{gl}(r, \mathbb{C}).$$

**Proposition 2.1.** (i) The equations of motion of the system with Hamiltonian (2.1) are

$$(2.2) \quad \begin{cases} dX/dt = Y^{k-1} v \alpha w + Y^{k-2} v \alpha w Y + \dots + v \alpha w Y^{k-1} \\ dY/dt = 0 \\ dv/dt = Y^k v \alpha \\ dw/dt = -\alpha w Y^k. \end{cases}$$

(ii) The Poisson brackets of the Hamiltonians (2.1) are given by

$$\{J_{k, \alpha}, J_{l, \beta}\} = J_{k+l, [\alpha, \beta]}.$$

The right hand side of the first equation in (2.2) is interpreted to be  $v \alpha w$  if  $k = 1$ , and 0 if  $k = 0$ .

In most cases it is easy to write down the solution to the equations (2.2) (see the next subsection). It is clear even without any calculations that the flows of these equations are *complete* (that is, they exist for all complex  $t$ ). Indeed, since  $Y$  is constant, the equations for the entries of  $v$  and  $w$  are linear with constant coefficients, so their solutions are combinations of polynomials and exponentials; the entries in  $X$  are therefore of the same kind. As Gibbons and Hermesen point out, the commutation relations in (ii) are the same as those in the Lie algebra of polynomial loops in  $\mathfrak{gl}(r)$  (let the loop  $z \mapsto \alpha z^k$  correspond to  $J_{k, \alpha}$ ). These

observations suggest that the flows of the Hamiltonians (2.1) should fit together to give an action on each  $\mathcal{C}_n(r)$  of the group  $\Gamma$  of holomorphic functions from  $\mathbb{C}$  to  $GL_r(\mathbb{C})$ . As we shall see next, on the subspace  $\mathcal{C}^d$  where  $Y$  has distinct eigenvalues, we can write down this action explicitly.

**Solution of the equations of motion on  $\mathcal{C}^d$ .** Let  $Y = \text{diag}(\lambda_1, \dots, \lambda_n)$ : then the equations (2.2) for the rows  $v_i$  of  $v$  and the columns  $w_j$  of  $w$  read

$$dv_i/dt = \lambda_i^k v_i \alpha, \quad dw_j/dt = -\alpha w_j \lambda_j^k,$$

with solution

$$(2.3) \quad v_i(t) = v_i(0) \exp(\lambda_i^k \alpha t), \quad w_j(t) = \exp(-\lambda_j^k \alpha t) w_j(0).$$

So the diagonal entries in  $X$  satisfy

$$dX_{ii}/dt = k \lambda_i^{k-1} v_i(0) \alpha w_i(0),$$

with solution

$$(2.4) \quad X_{ii} = X_{ii}(0) + k \lambda_i^{k-1} v_i(0) \alpha w_i(0) t,$$

while for  $i \neq j$  we have

$$\begin{aligned} dX_{ij}/dt &= (\lambda_i^{k-1} + \lambda_i^{k-2} \lambda_j + \dots + \lambda_j^{k-1}) (v \alpha w)_{ij} \\ &= \frac{\lambda_i^k - \lambda_j^k}{\lambda_i - \lambda_j} v_i(0) \alpha \exp\{(\lambda_i^k - \lambda_j^k) \alpha t\} w_j(0) \\ &= \frac{d}{dt} [v_i(0) \exp\{(\lambda_i^k - \lambda_j^k) \alpha t\} w_j(0)] / (\lambda_i - \lambda_j). \end{aligned}$$

So the solution is

$$(2.5) \quad X_{ij}(t) = (\text{constant}) + v_i(t) w_j(t) / (\lambda_i - \lambda_j)$$

(as we could have seen at once from the fact that  $[X, Y] + v w$  is constant). So far the calculation has been valid on the whole of  $V$ ; but if we are in  $\mathcal{C}$ , then  $X_{ij}(0) = v_i(0) w_j(0) / (\lambda_i - \lambda_j)$ , so the constant in (2.5) is zero. We can reformulate that remark as follows.

**Proposition 2.2.** *Let  $(X, Y; v, w) \in \mathcal{C}^d$ , with  $Y = \text{diag}(\lambda_1, \dots, \lambda_n)$ . If  $\gamma(z) \in \Gamma$ , define  $(X, Y; v, w) \circ \gamma$  by the formulae*

$$(2.6) \quad \begin{cases} (v \circ \gamma)_i &= v_i \gamma(\lambda_i) \\ (w \circ \gamma)_j &= \gamma(\lambda_j)^{-1} w_j \\ (X \circ \gamma)_{ii} &= X_{ii} + v_i \gamma'(\lambda_i) \gamma(\lambda_i)^{-1} w_i \\ (X \circ \gamma)_{ij} &= (v \circ \gamma)_i (w \circ \gamma)_j (\lambda_i - \lambda_j)^{-1} \\ Y \circ \gamma &= Y. \end{cases}$$

*Then this defines a right action of  $\Gamma$  on  $\mathcal{C}^d$ , and the trajectory of the 1-parameter subgroup  $\{\exp(\alpha z^k t)\}$  is the solution to the equations of motion (2.2).*

We remark that if  $Y$  is diagonal with repeated eigenvalues, the equations (2.2) are still easier to solve. In section 5 we shall meet the extreme case where  $Y$  is a

scalar matrix (in that case we have  $vw = -I$ , so it can happen only if  $n \leq r$ ). If  $Y = \lambda I$ , the equations (2.6) for the action of  $\Gamma$  simplify to

$$(2.7) \quad \begin{cases} v \circ \gamma = v\gamma(\lambda) \\ w \circ \gamma = \gamma(\lambda)^{-1}w \\ X \circ \gamma = X + v\gamma'(\lambda)\gamma(\lambda)^{-1}w \\ Y \circ \gamma = Y. \end{cases}$$

**The action of scalars.** We denote by  $\Gamma_{sc} \subseteq \Gamma$  the subgroup of scalar-valued maps, that is, those of the form  $\gamma(z) = e^{p(z)}I$  where  $p$  is an entire function. It is easy to write down how  $\Gamma_{sc}$  acts on  $\mathcal{C}$ . Note first that on the subspace  $\mu^{-1}(-I)$  of  $V$ , we have

$$\begin{aligned} \text{tr } Y^k vw &= \text{tr} \{Y^k(-[X, Y] - I)\} \\ &= \text{tr} \{[Y, Y^k X] - Y^k\} \\ &= -\text{tr } Y^k. \end{aligned}$$

Thus the Hamiltonians  $J_{k,I}$  and  $-\text{tr } Y^k$  induce the same flows on our space  $\mathcal{C}$  (on  $V$  the flows differ by the action of  $g(t) := \exp(-tY^k)$ ). The equations of motion are now simply  $dX/dt = -kY^{k-1}$ , other variables constant, with solution

$$X(t) = X(0) - kY^{k-1}t, \quad Y, v, w \text{ constant}.$$

Note particularly that the 1-parameter subgroup  $\{\exp(xz)I\}$  acts on  $\mathcal{C}$  by

$$(2.8) \quad X \mapsto X - xI, \quad Y, v, w \text{ constant}.$$

The matrices  $X - xI$  for different values of  $x$  are never conjugate to each other (because their eigenvalues are different); thus this subgroup acts *freely* on each space  $\mathcal{C}_n$  (for  $n > 0$ ).

Generalizing slightly, we get the following formula for the action of  $\Gamma_{sc}$ .

**Proposition 2.3.** *Let  $\gamma(z) := e^{p(z)}I \in \Gamma_{sc}$ . Then the action of  $\gamma$  on  $\mathcal{C}$  is given by*

$$(2.9) \quad (X, Y; v, w) \circ \gamma = (X - p'(Y), Y; v, w)$$

*just as in the case  $r = 1$ .*

*Remark.* There are other cases where we can write down the solution to the equations (2.2) for any  $Y$ : for example, note that

$$\frac{d}{dt}(v\alpha w) = Y^k v\alpha^2 w - v\alpha^2 w Y^k = [Y^k, v\alpha^2 w],$$

so if  $\alpha^2 = 0$  then  $dX/dt$  is again constant. The case where  $\alpha$  is a projection ( $\alpha^2 = \alpha$ ) is another easy one. It is annoying that in general there does not seem to be one simple formula for the action of  $\Gamma$ : maybe we need to check that this action really exists.

## 3. SOME GRASSMANNIANS

We start off in the spirit of [SW]. Let  $H$  be the Hilbert space of  $L^2$  functions  $f : S^1 \rightarrow \mathbb{C}^r$ : we shall regard the elements of  $H$  as *row* vectors  $f = (f_0, \dots, f_{r-1})$  of scalar valued functions. Let  $\text{Gr}$  be the restricted Grassmannian associated to the usual polarization  $H = H_+ \oplus H_-$ . The group  $\Gamma$  of entire loops  $g : \mathbb{C} \rightarrow GL(r, \mathbb{C})$  acts in the obvious way (on the right) on  $H$  and thence on  $\text{Gr}$ . If  $W \in \text{Gr}$ , we have the *Baker function*  $\psi_W(g, z)$ : it is the unique (meromorphic)  $r \times r$  matrix valued function on  $\Gamma \times S^1$  such that

- (1) it has the form  $\psi_W(g, z) = (\mathbf{I} + \sum_1^\infty a_i(g)z^{-i})g(z)$ ;
- (2) each row of  $\psi_W(g, z)$  belongs to  $W$  for each  $g \in \Gamma$ .

Thus the  $i^{\text{th}}$  row of  $\psi_W$  is just the inverse image of the  $i^{\text{th}}$  basis vector  $e_i \in \mathbb{C}^r$  under the projection  $Wg^{-1} \rightarrow H_+$ . The Baker function is singular at the points  $g \in \Gamma$  where  $Wg^{-1}$  is not in the *big cell*, that is, where this projection is not an isomorphism. We write  $\tilde{\psi}_W(g, z)$  for the term  $\mathbf{I} + \dots$  in (1) above, so that  $\psi_W(g, z) = \tilde{\psi}_W(g, z)g(z)$ . Note the formula

$$(3.1) \quad \tilde{\psi}_{W\gamma}(g, z) = \tilde{\psi}_W(g\gamma^{-1}, z) \quad (g, \gamma \in \Gamma),$$

which follows directly from the definitions. If we restrict  $g$  to run over the 1-parameter subgroup  $\{\exp(xz)\mathbf{I}\}$  of  $\Gamma$ , we obtain the *stationary Baker function*

$$\psi_W(x, z) := \tilde{\psi}_W(\exp(xz), z)e^{xz}.$$

In contrast to the case  $r = 1$ , the stationary Baker function may not exist, because the flow  $W \mapsto We^{xz}$  may stay entirely outside the big cell; also, even if it does exist, it may be independent of  $x$  (see section 5 for examples).

In the above discussion, the unit circle  $S^1$  could be replaced by a larger circle, and we can form the union of the resulting Grassmannians as the circles get larger. Changing notation, from now on we shall denote this union by  $\text{Gr}$ .

In this paper we are mainly concerned with the much smaller Grassmannian  $\text{Gr}^{\text{rat}}$ : it consists of the subspaces  $W \in \mathbb{C}(z)^r$  such that

- (1)  $p(z)\mathbb{C}[z]^r \subseteq W \subseteq q(z)^{-1}\mathbb{C}[z]^r$  for some polynomials  $p, q$ ;
- (2) the codimension of  $W$  in  $q(z)^{-1}\mathbb{C}[z]^r$  is  $r \deg(q)$ .

As in [W2] or [BW2], a suitable closure  $\overline{W}$  of  $W$  then belongs to  $\text{Gr}$ , and we can recover  $W$  as the intersection  $W = \overline{W} \cap \mathbb{C}(z)^r$ . In this way we may regard  $\text{Gr}^{\text{rat}}$  as a subspace of  $\text{Gr}$ ; the action of  $\Gamma$  on  $\text{Gr}$  clearly preserves this subspace, because we have  $p(z)H \subseteq \overline{W} \subseteq q(z)^{-1}H$ . For each  $\lambda \in \mathbb{C}$  we have the subspace  $\text{Gr}_\lambda$  of  $\text{Gr}^{\text{rat}}$  consisting of those  $W$  for the polynomials  $p$  and  $q$  can be chosen to be powers of  $z - \lambda$ .

It will be convenient to work momentarily with the larger space  $\mathbb{G}^{\text{rat}}$  of all *fat* subspaces of  $\mathbb{C}(z)^r$  (that is, subspaces satisfying just the first condition in the definition of  $\text{Gr}^{\text{rat}}$ ). We denote by  $\mathbb{G}^{\text{ad}}$  the set of all *primary decomposable* subspaces in  $\mathbb{G}^{\text{rat}}$  [definition to be discussed elsewhere]. Let  $W \in \mathbb{G}^{\text{rat}}$ . We define  $M_W \subseteq \mathbb{C}(z)[\partial_z]^r$  to consist of all operators  $D = (D_1, \dots, D_r)$  such that<sup>1</sup>  $D \cdot \mathbb{C}[z] \subseteq W$  (that is, for each polynomial  $p(z)$  the vector  $(D_1.p, \dots, D_r.p)$  is in  $W$ ). Clearly,  $M_W$  is a fat right sub- $A$ -module of  $\mathbb{C}(z)[\partial_z]^r$  (we recall again from

<sup>1</sup>It would be more proper to write  $\mathbb{C}[z] \star D \subseteq W$  since  $M_W$  is just the space  $\mathcal{D}(\mathbb{C}[z], W)$  studied in section 7 below.

[BGK] that  $M$  is fat if  $pA^r \subseteq M \subseteq q^{-1}A^r$  for some polynomials  $p$  and  $q$ ). The following is proved in [CH] for  $r = 1$  and in [BGK] in general.

**Theorem 3.1.** *The map  $W \mapsto M_W$  gives a bijection from  $\mathbb{G}^{\text{ad}}$  to the set of fat right sub- $A$ -modules of  $\mathbb{C}(z)[\partial_z]^r$ . The inverse map sends  $M$  to  $M \cdot \mathbb{C}[z]$ .*

Now let  $L_W \subseteq \mathbb{C}(z)^r$  be the space of all leading coefficients of the operators in  $M_W$ . Clearly,  $L_W$  is a lattice in  $\mathbb{C}(z)^r$ , that is, a sub- $\mathbb{C}[z]$ -module of full rank  $r$ .

**Definition 3.2.** The adelic Grassmannian  $\text{Gr}^{\text{ad}}$  is the subset of those  $W \in \mathbb{G}^{\text{ad}}$  such that  $L_W = \mathbb{C}[z]^r$ .

This definition (though not the notation) is taken from [BGK]: it is probably the most important part of that paper. In the case  $r = 1$ , the lattice  $L_W$  is just a (principal) fractional ideal  $(q) \subset \mathbb{C}(z)$ ; the group  $\mathbb{C}(z)^\times$  acts on  $\mathbb{G}^{\text{ad}}$ , and we can view  $\text{Gr}^{\text{ad}}$  as the quotient space. For  $r > 1$  this point of view is not possible: the group  $GL_r(\mathbb{C}(z))$  acts (compatibly) on  $\mathbb{G}^{\text{ad}}$  and on the lattices in  $\mathbb{C}(z)^r$ , however,  $\mathbb{G}^{\text{ad}}/GL_r(\mathbb{C}(z))$  is not  $\text{Gr}^{\text{ad}}$ , but the quotient  $\text{Gr}^{\text{ad}}/GL_r(\mathbb{C}[z])$ .

*Remark.* As observed in [BGK], the larger group  $GL_r(\mathbb{C}(z)[\partial_z])$  acts compatibly on fat submodules and on  $\text{Gr}^{\text{ad}}$ . According to Stafford, torsion-free  $A$ -modules of rank  $> 1$  are free, so this action is transitive. That leads to an identification

$$\mathbb{G}^{\text{ad}} \simeq GL_r(\mathbb{C}(z)[\partial_z])/GL_r(A) \quad (r > 1),$$

but I do not know what use that is.

Another thing to note is that for  $r = 1$  every point of the spaces  $\text{Gr}_\lambda$  belongs to  $\text{Gr}^{\text{ad}}$ , but for  $r > 1$  that is not so. One reason is provided by the following lemma.

**Lemma 3.3.** *Suppose that  $W \in \mathbb{G}^{\text{ad}}$  is such that  $zW \subseteq W$ . Then  $W = L_W$ .*

*Proof.* The condition  $zW \subseteq W$  implies that  $zM_W \subseteq M_W$ . By construction,  $M_W z \subseteq M_W$ , so  $M_W$  is stable under  $\text{ad } z$ . But  $(\text{ad } z)(p\partial_z^k) = -kp\partial_z^{k-1}$ ; it follows that  $M_W$  contains the leading coefficients of all its operators, that is,  $M_W = L_W A$ . By Theorem 3.1,  $W = M_W \cdot \mathbb{C}[z] = L_W$ .  $\square$

**Corollary 3.4.** *The only point  $W \in \text{Gr}^{\text{ad}}$  such that  $zW \subseteq W$  is the base-point  $\mathbb{C}[z]^r$ .*

#### 4. THE MAP FROM $\mathcal{C}$ TO $\text{Gr}^{\text{ad}}$

We fix  $n$  distinct complex numbers  $\lambda_i$ , another  $n$  complex numbers  $\alpha_i$ , and  $n$  pairs of vectors  $(v_i, w_i)$  with  $v_i w_i = -1$  (as above, the  $v_i$  are  $1 \times r$  vectors and the  $w_i$  are  $r \times 1$ ). To these data we assign the following space  $W \in \text{Gr}^{\text{rat}}(r)$ : a vector-valued rational function  $f$  belongs to  $W$  if

- (1) it is regular everywhere except at the points  $\lambda_i$  and at  $\infty$ ;
- (2) it has (at most) a simple pole at each point  $\lambda_i$ ;
- (3) if  $f = \sum_{k=-1}^{\infty} f_k^{(i)}(z - \lambda_i)^k$  is the Laurent expansion of  $f$  near  $\lambda_i$ , then
  - (a)  $f_{-1}^{(i)}$  is a scalar multiple of  $v_i$ ; and
  - (b)  $(f_0^{(i)} + \alpha_i f_{-1}^{(i)})w_i = 0$ .

Note that in the case  $r = 1$ , the condition 3(a) is vacuous, and 3(b) simply says that  $f_0^{(i)} + \alpha_i f_{-1}^{(i)} = 0$ , so we recover the space  $W(\boldsymbol{\lambda}, \boldsymbol{\alpha})$  of [W2].

Let us calculate the Baker function of  $W$ . Conditions (1) and (2) above imply that it has the form

$$\psi_W(g, z) = \left( \mathbf{I} + \sum_1^n \frac{A_i(g)}{z - \lambda_i} \right) g(z) ,$$

where  $g \in \Gamma$ , and the  $r \times r$  matrices  $A_i(g)$  are determined by the requirement that the rows of  $\psi_W$  all belong to  $W$ . In the Laurent expansion of  $\psi_W$  around  $z = \lambda_i$ , the residue is  $A_i(g)g(\lambda_i)$ , and the constant term is

$$A_i(g)g'(\lambda_i) + \left\{ \mathbf{I} + \sum_{j \neq i} \frac{A_j(g)}{\lambda_i - \lambda_j} \right\} g(\lambda_i) .$$

So condition 3(a) above says that  $A_i(g)g(\lambda_i) = a_i(g)v_i$  for some  $r \times 1$  vectors  $a_i$ ; and the condition 3(b) reads

$$\left\{ A_i(g)g'(\lambda_i) + \left[ \mathbf{I} + \sum_{j \neq i} \frac{A_j(g)}{\lambda_i - \lambda_j} + \alpha_i A_i(g) \right] g(\lambda_i) \right\} w_i = 0 .$$

Combining these equations and using  $v_i w_i = -1$ , we get

$$a_i(g) [v_i g(\lambda_i)^{-1} g'(\lambda_i) w_i - \alpha_i] + \sum_{j \neq i} a_j(g) \frac{v_j g(\lambda_j)^{-1} g(\lambda_i) w_i}{\lambda_i - \lambda_j} = -g(\lambda_i) w_i .$$

If we set

$$(4.1) \quad \begin{cases} v_i(g) := v_i g(\lambda_i)^{-1} \\ w_i(g) := g(\lambda_i) w_i \end{cases}$$

that becomes

$$a_i(g) [v_i g(\lambda_i)^{-1} g'(\lambda_i) w_i - \alpha_i] + \sum_{j \neq i} a_j(g) \frac{v_j(g) w_j(g)}{\lambda_i - \lambda_j} = -w_i(g) ;$$

or, finally,

$$a(g) X(g) = w(g) ,$$

where  $a(g)$  and  $w(g)$  are the  $r \times n$  matrices with columns  $a_i(g)$  and  $w_i(g)$ , and  $X(g)$  is the  $n \times n$  matrix with entries

$$(4.2) \quad \begin{aligned} X_{ii}(g) &= \alpha_i - v_i g(\lambda_i)^{-1} g'(\lambda_i) w_i \\ X_{ij}(g) &= \frac{v_i(g) w_j(g)}{\lambda_i - \lambda_j} \quad \text{for } i \neq j . \end{aligned}$$

Let  $Y := \text{diag}(\lambda_1, \dots, \lambda_n)$ ; then we have

$$\begin{aligned} \sum_1^n \frac{A_i(g)}{z - \lambda_i} &= \sum_1^n a_i(g) v_i(g) (z - \lambda_i)^{-1} \\ &= a(g) (z\mathbf{I} - Y)^{-1} v(g) \\ &= w(g) X(g)^{-1} (z\mathbf{I} - Y)^{-1} v(g) , \end{aligned}$$

where  $v(g)$  is the  $n \times r$  matrix with rows  $v_i(g)$ . So finally we get the formula

$$(4.3) \quad \psi_W(g, z) = \left\{ \mathbf{I} + w(g) X(g)^{-1} (z\mathbf{I} - Y)^{-1} v(g) \right\} g(z) .$$

It is now clear how to define the map  $\beta : \mathcal{C}^d \rightarrow \text{Gr}^{\text{ad}}$ . Let  $(X, Y; v, w) \in \mathcal{C}^d$ , with  $Y = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $X_{ii} = \alpha_i$ ; then  $\beta$  maps  $(X, Y; v, w)$  to the space  $W \in \text{Gr}^{\text{ad}}$  specified by the conditions (1)–(3) at the start of this section.

**Proposition 4.1.** *This map  $\beta : \mathcal{C}^d \rightarrow \text{Gr}^{\text{ad}}$  is  $\Gamma$ -equivariant.*

*Proof.* The quadruple  $(X(g), Y; v(g), w(g))$  given by the formulae (4.1) and (4.2) is just  $(X, Y; v, w) \circ g^{-1}$ , where  $\circ$  refers to the action (2.6). So it follows from (4.3) that  $\beta$  maps  $(X, Y; v, w) \circ \gamma$  to the space with reduced Baker function  $\tilde{\psi}_W(g\gamma^{-1}, z)$ ; by (3.1), that is  $W\gamma$ , as claimed.  $\square$

If we restrict  $g$  to run over the 1-parameter subgroup  $\{\exp(xz)I\}$  of  $\Gamma$ , the formula (4.3) simplifies to

$$(4.4) \quad \tilde{\psi}_W(x, z) = I + w(xI + X)^{-1}(zI - Y)^{-1}v.$$

We use this formula to define  $\beta : \mathcal{C} \rightarrow \text{Gr}^{\text{ad}}$  in general; that is,  $\beta$  maps  $(X, Y; v, w)$  to the point  $W \in \text{Gr}^{\text{ad}}$  whose stationary Baker function is given by (4.4). The main result of these notes should be the following.

**Theorem 4.2.** *As for  $r = 1$ , the maps  $\beta$  give a  $\Gamma$ -equivariant bijection*

$$\bigsqcup_{n \geq 0} \mathcal{C}_n(r) \rightarrow \text{Gr}^{\text{ad}}(r).$$

I shall not prove that here; but see Proposition 7.4 below, which takes care of the first new point (involving the definition of  $\text{Gr}^{\text{ad}}$ ). There are two possible ways to prove Theorem 4.2. One is to imitate the arguments of [W2] (Shiota's lemma might give trouble). The other is to show that  $\beta$  is the same as the bijection constructed in [BGK] (indirectly, passing through bundles over the noncommutative quadric). For  $r = 1$  the only way we know to see that relies on having a group acting transitively on each  $\mathcal{C}_n$ , so we would need something like Conjecture 8.1 below. Probably both ways are worth doing, but neither is really satisfactory. Even for  $r = 1$ , the definition of  $\beta$  is hard to understand, and has the capital defect that it is not obvious that it makes sense locally (in  $z$ ). And no direct definition at all is known for its inverse. A more intelligent approach is needed...

*Remark.* In the case  $r = 1$ , the equation (4.4) can be rewritten in the form

$$(4.5) \quad \tilde{\psi}_W(x, z) = \det \{I - (zI - Y)^{-1}(xI + X)^{-1}\}$$

(see [W2], p. 16) The discrepancies between the formulae (4.4) and (4.5) and the similar ones in [W2] are due partly to changes of notation, partly to errors in [W2] (these errors are more difficult to make in the case  $r > 1$ ). To recover the exact formulae of [W2] we have to replace  $(X, Y; v, w)$  by  $(X^t, -Y^t; w^t, v^t)$ , then change the sign of one of  $v$  or  $w$ , and restore the notation  $Z$  instead of  $Y$ .

## 5. MORE EXAMPLES

This section gathers together various calculations, some of them superfluous.



**Some points of  $\text{Gr}_0$ .** Here we shall look at some of the points  $W \in \text{Gr}_0$  which satisfy  $z\mathbb{C}[z]^r \subseteq W \subseteq z^{-1}\mathbb{C}[z]^r$ . Such  $W$  form a space isomorphic to the Grassmannian  $\text{Gr}(r, 2r)$  of  $r$ -dimensional subspaces of  $\mathbb{C}^{2r}$ . If a typical element of  $W$  has Laurent expansion  $f = f_{-1}z^{-1} + f_0 + O(z)$ , then the various  $W \in \text{Gr}(r, 2r)$  can be specified by imposing conditions of the form

$$f_{-1}A + f_0B = 0,$$

where the  $r \times 2r$  matrix  $(A|B)$  has full rank  $r$ .

First, consider one of the “big cells” where  $B$  is invertible; we may as well suppose that  $B$  is the identity, so the corresponding  $W$  consists of functions  $f$  such that  $f_{-1}A + f_0 = 0$ . A calculation like the one in the preceding section (but much easier) shows that the (reduced) Baker function of  $W$  is

$$(5.1) \quad \tilde{\psi}_W(g, z) = \text{I} - g(0)\{g(0)^{-1}g'(0) + A\}^{-1}z^{-1}g(0)^{-1};$$

in particular, the stationary Baker function is

$$\tilde{\psi}_W(x, z) = \text{I} - (x\text{I} + A)^{-1}z^{-1}.$$

Comparing with the formulae (2.7), (4.3) and (4.4), we see that  $W$  is the image under  $\beta$  of the quadruple

$$(X, Y; v, w) = (A, 0; \text{I}, -\text{I}) \in \mathcal{C}_r(r).$$

Now let us look at the “opposite” big cell where  $A$  is the identity, so that  $W$  consists of functions  $f$  such that  $f_{-1} + f_0B = 0$ . Calculating again as above, we find this time that the Baker function is

$$(5.2) \quad \tilde{\psi}_W(g, z) = \text{I} - g(0)B\{\text{I} + g(0)^{-1}g'(0)B\}^{-1}z^{-1}g(0)^{-1}.$$

If  $B = A^{-1}$  is invertible, then of course we recover the previous formula (5.1). Let us look at the opposite extreme. For  $B = 0$  we just get the base-point  $W = \mathbb{C}[z]^r$ , so the smallest interesting case is where  $B$  has rank 1, say  $B = ab$ , where  $a$  and  $b$  are column and row vectors of length  $r$ . Using the formula

$$(\text{I} - ab)^{-1} = \text{I} + ab(1 - ba)^{-1}$$

for the inverse of an elementary matrix (note that  $ba$  is a scalar) we can simplify (5.2) to get

$$(5.3) \quad \tilde{\psi}_W(g, z) = \text{I} - g(0) \frac{ab}{1 + \beta(g)} z^{-1}g(0)^{-1},$$

where we have set  $\beta(g) := b g(0)^{-1}g'(0)a$ .

Now observe that the condition  $f_{-1} + f_0ab = 0$  defining  $W$  implies that (i)  $f_{-1}$  is a scalar multiple of  $b$ ; and (ii)  $f_{-1}a = -f_0aba = -(ba)f_0a$ . If  $ba \neq 0$ , then (ii) can be written as  $(f_0 + \alpha f_{-1})a = 0$ , where  $\alpha := (ba)^{-1}$ . Thus  $W$  corresponds to the quadruple  $(X, Y; v, w) = (\alpha, 0; b, -\alpha a) \in \mathcal{C}_1(r)$ . On the other hand, if  $ba = 0$ , then  $\beta(e^{xz}) \equiv 0$ , so the *stationary* Baker function of  $W$  is

$$\tilde{\psi}_W(x, z) = \text{I} - ab z^{-1}.$$

Notice that it is independent of  $x$ ! From the formulae (2.8) and (4.4) we see that this can never happen if  $W$  comes from a point of  $\mathcal{C}$ , so these spaces  $W$  provide examples of points of  $\text{Gr}_0$  that do not belong to  $\beta(\mathcal{C})$ . However, we knew that already, because  $ba = 0$  implies  $zW \subseteq W$  (cf. Lemma 3.3).

**A direct calculation of  $L_W$ .** Let us look at the example above in the case  $a = (1, 0)^t$ ,  $b = (0, 1)$ , so that  $W \in \text{Gr}_0(2)$  is the space of vectors  $f$  with Laurent expansion of the form

$$f = (0, c)z^{-1} + (c, d) + O(z) .$$

**Lemma 5.1.** *The right  $A$ -module  $\mathcal{D}(\mathbb{C}[z], W)$  is (freely) generated by  $(1, z^{-1})$  and  $(0, 1)$ .*

It follows that the lattice in  $\mathbb{C}(z)^2$  associated to  $W$  has the same generators, confirming Lemma 3.3.

To check Lemma 5.1, it is slightly easier to work with the space of polynomial functions  $V := zW$ . A vector  $f = (f_0(z), f_1(z))$  belongs to  $V$  if and only if we have

$$(5.4) \quad f_0(0) = 0 \quad \text{and} \quad f_1(0) = f'_0(0) .$$

Let  $D := (E, F) \in \mathcal{D}(\mathbb{C}[z], V)$ . Then the first condition in (5.4) is equivalent to  $(E.p)(0) = 0$  for all polynomials  $p$ ; that is,  $E.\mathbb{C}[z] \subseteq z\mathbb{C}[z]$ , or  $E \in zA$ . Set  $E = zP$  (where  $P \in A$ ). Then  $E.p = z(P.p)$ , whence  $(E.p)' = P.p + z(P.p)'$ , so  $(E.p)'(0) = (P.p)(0)$ . So the second condition in (5.4) is equivalent to

$$(F - P).\mathbb{C}[z] \subseteq z\mathbb{C}[z] ,$$

so  $F - P \in zA$ , say  $F - P = zQ$ . Thus  $\mathcal{D}(\mathbb{C}[z], V)$  consists of all operators of the form

$$D = (zP, P + zQ) = (z, 1)P + (0, z)Q \quad \text{for some } P, Q \in A .$$

In other words, the right  $A$ -module  $\mathcal{D}(\mathbb{C}[z], V) \subseteq A^2$  is generated by  $(z, 1)$  and  $(0, z)$ . Lemma 5.1 follows.

**Outside the big cell.** Here we give an example where the *stationary* Baker function does not exist. That happens when the flow  $W \mapsto We^{xz}$  on  $\text{Gr}(r)$  stays outside the big cell for all  $x$ . By Proposition 8.6 in [SW], that is impossible for  $r = 1$ . But if we identify  $H(r)$  with  $H := H(1)$  via the “interleaving Fourier series” isomorphism (see [SW], p. 14), then multiplication by  $z$  on  $H(r)$  corresponds to multiplication by  $z^r$  on  $H$ , so the  $x$ -flow on  $\text{Gr}(r)$  corresponds to the  $r^{\text{th}}$  KP flow  $W \mapsto We^{t_r z^r}$  on  $\text{Gr}(1)$ . Any of these flows (for  $r > 1$ ) can have orbits that do not meet the big cell.

Let us take  $r = 2$ . The simplest example I know where the  $t_2$ -flow on  $\text{Gr}(1)$  lies outside the big cell starts at the point  $H_S$  where  $S = \{-3, -1; 2, 3, \dots\}$ . The  $\tau$ -function is  $(-24 \text{ times})$  the Schur function of the partition  $(3, 2)$ , namely

$$(5.5) \quad \tau_S(\mathbf{t}) = t_1^5 - 4t_2t_1^3 - 12t_3t_1^2 + (12t_2^2 + 24t_4)t_1 - 24t_2t_3 .$$

The corresponding space  $W \in \text{Gr}_0(2)$  consists of all (vector-valued) functions of the form  $f(z) = f_{-2}z^{-2} + f_{-1}z^{-1} + \dots$  satisfying the four conditions

- the first entry in  $f_{-2}$  vanishes;
- the first entry in  $f_{-1}$  vanishes;
- both entries in  $f_0$  vanish.

Let us see what happens if we try to calculate the stationary Baker function of this  $W$ . It must have the form

$$\psi_W(x, z) = (I + A(x)z^{-1} + B(x)z^{-2})e^{xz} .$$

The coefficients of  $z^{-2}$ ,  $z^{-1}$  and 1 in  $\psi_W(x, z)$  are (respectively)

$$B(x); \quad A(x) + xB(x); \quad \text{and} \quad I + xA(x) + \frac{1}{2}x^2B(x) .$$

The conditions above for the rows of  $\psi_W$  to belong to  $W$  say that the first column of  $B(x)$  vanishes; then that the first column of  $A(x)$  vanishes; and finally that the first column of  $I$  vanishes!

*Remark 5.2.* Let us recall that (in the case  $r = 1$ ) the roots of the polynomial  $\tau_W(t_1, 0, 0, \dots)$  are the eigenvalues of the matrix  $X$  in the pair  $(X, Y) \in \mathcal{C}$  corresponding to  $W$ , and that  $W$  is outside the big cell exactly when  $X$  is singular, that is when  $\tau_W(0, 0, 0, \dots) = 0$ . The  $k^{\text{th}}$  KP flow just translates the variable  $t_k$  in  $\tau_W$ . In our example (5.5), we have  $\tau_S(0, t_2, 0, \dots) \equiv 0$ ; that means indeed that the  $t_2$ -flow stays outside the big cell. This  $\tau$ -function is quite interesting from another point of view: we have  $\tau_S(t_1, 0, t_3, 0, \dots) = t_1^5 - 12t_3t_1^2$ , so the  $t_3$ -flow stays inside the “bad” set where  $\tau_W(t_1, 0, 0, \dots)$  has a multiple root.

## 6. TRANSPOSING MATRICES

The appearance of a transpose in the formula (7.1) below forces us to notice some elementary points, which we review in this section

Let  $R$  be a ring: we denote the multiplication in  $R$  by juxtaposition and the opposite multiplication by  $\star$ , so that we have  $p \star q = qp$ . In what follows,  $D, E, \dots$  will denote matrices with entries in  $R$  (not necessarily square, but of sizes such that the products we write down are defined). The transpose of a matrix  $D$  is written  $D^t$ . If  $R$  is commutative, we have the formula  $(DE)^t = E^t D^t$ : in general that becomes  $(D \star E)^t = E^t D^t$ , or, equivalently

$$(6.1) \quad D \star E = (E^t D^t)^t .$$

Here  $D \star E$  denotes the usual product of matrices, but using the  $\star$ -multiplication on the scalar entries.

If  $b$  is an anti-automorphism of  $R$ , then we have a formula not using  $\star$ , namely

$$(6.2) \quad b(DE)^t = b(E)^t b(D)^t .$$

As a special case, let  $A$  be an invertible (square) matrix over  $R$ . Then applying the last rule to the formula  $AA^{-1} = I$ , we get  $b(A^{-1})^t b(A)^t = I$ , that is

$$(6.3) \quad [b(A)^t]^{-1} = b(A^{-1})^t .$$

Now let  $\mathcal{F}$  be a left  $R$ -module; in what follows  $\varphi$  will denote a matrix (not necessarily square) with entries in  $\mathcal{F}$ . Matrices over  $R$  act on the left of  $\varphi$  in an obvious way. Regarding  $\mathcal{F}$  as a right module over the ring opposite to  $R$ , we make them act also on the right, setting

$$(6.4) \quad \varphi \star D := (D^t \cdot \varphi^t)^t ,$$

where  $\cdot$  denotes the given left action. We then have the rule

$$(6.5) \quad (\varphi \star D) \star E = \varphi \star (D \star E) .$$

*Remark.* In our application  $R$  will be a ring of differential operators,  $\mathcal{F}$  a module of differentiable functions.

## 7. BISPECTRALITY

If in the formula (4.4) we interchange  $x$  and  $z$  and take the matrix transpose, we get

$$(7.1) \quad \tilde{\psi}_W(z, x)^t = I + v^t(xI - Y^t)^{-1}(zI + X^t)^{-1}w^t.$$

This has the same form as (4.4), but with  $(X, Y; v, w)$  replaced by

$$(7.2) \quad b(X, Y; v, w) := -(Y^t, X^t; w^t, v^t).$$

Clearly,  $b$  is an involution on  $\mathcal{C}$ ; we use the same symbol  $b$  to denote the induced involution on  $\text{Gr}^{\text{ad}}$ , so that we have the formula

$$(7.3) \quad \psi_{b(W)}(x, z) = \psi_W(z, x)^t.$$

Note that the corresponding formula for the operator  $K_W$  is

$$(7.4) \quad K_{b(U)} = b(K_U)^t,$$

where on the right  $b$  is the anti-automorphism which interchanges  $x$  and  $\partial_x$ .

Now fix two integers  $r$  and  $s$ ; let  $U \in \text{Gr}^{\text{ad}}(r)$  and  $V \in \text{Gr}^{\text{ad}}(s)$ . We define  $\mathcal{D}(U, V)$  to be the set of all  $r \times s$  matrices  $D$  with entries in  $\mathbb{C}(z)[\partial_z]$  such that  $U \star D \subseteq V$ . The next proposition generalizes Proposition 8.2 in [BW2], and is proved exactly as in [BW2].

**Proposition 7.1.** *The following are equivalent.*

- (i)  $D \in \mathcal{D}(U, V)$ .
  - (ii) *There is an  $r \times s$  matrix  $\Theta(x)$  with entries in  $\mathbb{C}(x)[\partial_x]$  such that  $\psi_U(x, z) \star D(z) = \Theta(x) \cdot \psi_V(x, z)$ .*
  - (iii) *The  $r \times s$  matrix operator  $K_U b(D)(x) K_V^{-1}$  is differential.*
- Furthermore, the matrix  $\Theta$  is uniquely determined by the formula in (ii), and coincides with the operator  $K_U b(D)(x) K_V^{-1}$  in (iii).

**Proposition 7.2.** *The map  $D(z) \mapsto \Theta(z)^t$  is a bijection from  $\mathcal{D}(U, V)$  to  $\mathcal{D}(b(V), b(U))$ .*

*Proof.* Let  $D \in \mathcal{D}(U, V)$ . Interchanging  $x$  and  $z$  in the formula in (ii) above characterizing  $\Theta$ , we find that it is equivalent to

$$\psi_U(z, x) \star D(x) = \Theta(z) \cdot \psi_V(z, x).$$

By (7.3), that is the same as

$$\psi_{b(U)}(x, z)^t \star D(x) = \Theta(z) \cdot \psi_{b(V)}(x, z)^t.$$

By (6.4), this is equivalent to

$$(D(x)^t \cdot \psi_{b(U)}(x, z))^t = (\psi_{b(V)}(x, z) \star \Theta(z)^t)^t,$$

or, finally,

$$\psi_{b(V)}(x, z) \star \Theta(z)^t = D(x)^t \cdot \psi_{b(U)}(x, z).$$

By Proposition 7.1, this means that  $\Theta(z)^t \in \mathcal{D}(b(V), b(U))$ . Proposition 7.2 follows by symmetry.  $\square$

*Second proof.* Let us check that again, this time using the formula in (iii) of Proposition 7.1. Let  $D \in \mathcal{D}(U, V)$ , so that the operator  $\Theta := K_U b(D) K_V^{-1}$  is differential. Using the rules (6.2) and (6.3), we can calculate

$$b(\Theta)^t = b(K_V^{-1})^t D^t b(K_U)^t = K_{b(V)}^{-1} D^t K_{b(U)}.$$

Hence  $D^t = K_{b(V)} b(\Theta)^t K_{b(U)}^{-1}$ . Since  $D^t$  is differential, Proposition 7.1 shows again that  $\Theta^t \in \mathcal{D}(b(V), b(U))$ .  $\square$

Now let us set  $V = U$  in the above; we write  $\mathcal{D}(U)$  instead of  $\mathcal{D}(U, U)$ . By (6.5),  $\mathcal{D}(U)$  is an algebra with respect to the multiplication (6.1). If  $D \in \mathcal{D}(U)$  set  $B(D) := \Theta^t$  (where  $\Theta$  is as above).

**Proposition 7.3.** *The bijection  $B : \mathcal{D}(U) \rightarrow \mathcal{D}(b(U))$  is a anti-isomorphism of algebras.*

*Proof.* Calculate: we have

$$\begin{aligned} \psi_U \star (D_1 \star D_2) &= (\psi_U \star D_1) \star D_2 \\ &= (\Theta_1 \cdot \psi_U) \star D_2 \\ &= \Theta_1 \cdot (\psi_U \star D_2) \\ &= \Theta_1 \Theta_2 \cdot \psi_U . \end{aligned}$$

That means that

$$\begin{aligned} B(D_1 \star D_2) &= (\Theta_1 \Theta_2)^t \\ &= (B(D_1)^t B(D_2)^t)^t \\ &= B(D_2) \star B(D_1) , \end{aligned}$$

as claimed.  $\square$

As another application of Proposition 7.1, we prove the following fact, which should be a first step towards Theorem 4.2.

**Proposition 7.4.** *Let  $W \in \beta(\mathcal{C}) \subset \text{Gr}^{\text{rat}}$ . Then the lattice  $L_W$  in  $\mathbb{C}(z)^r$  is the base-point  $\mathbb{C}[z]^r$ .*

*Proof.* Let  $\mathcal{D} := \mathcal{D}(\mathbb{C}[z], W)$ . We have to show that

- (1) the leading coefficients of the operators in  $\mathcal{D}$  are all (vectors of) polynomials;
- (2) every polynomial  $p$  occurs as the leading coefficient of an operator in  $\mathcal{D}$ .

Since  $K_{\mathbb{C}[z]} = \text{I}$ , Proposition 7.1 tells us that  $D$  belongs to  $\mathcal{D}$  if and only if  $D$  and  $b(D)K_W^{-1}$  are both differential. Since

$$b(b(D)K_W^{-1})^t = b(K_W^{-1})^t D^t = K_{b(W)}^{-1} D^t ,$$

it is equivalent to say that  $D$  belongs to  $\mathcal{D}$  if and only if it is differential and its transpose has the form

$$D^t = K_{b(W)} P ,$$

where  $P$  is a *polynomial* operator (that is, all its coefficients are polynomials). Note first that the leading coefficient of  $D^t$  is the same as that of  $P$ , hence is a polynomial, which proves (1) above. To prove (2), we have to find enough polynomial operators  $P$  such that  $K_{b(W)} P$  is differential. For that, recall that if  $W = \beta(X, Y; v, w)$ , then

$$K_{b(W)} = \text{I} + v^t (x\text{I} - Y^t)^{-1} (\partial_x \text{I} + X^t)^{-1} w^t .$$

So if  $g(\lambda) := \det(\lambda\text{I} + X)$ , then  $K_{b(W)} g(\partial_x)$  is differential; more generally, for any  $p \in \mathbb{C}[z]^r$ ,  $K_{b(W)} p(x) g(\partial_x)$  is differential. This operator has leading coefficient  $p$ , and its transpose belongs to  $\mathcal{D}$  (since  $P := p(x) g(\partial_x)$  is a polynomial operator). That completes the proof.  $\square$

## 8. GROUPS

So far we have worked with the group  $\Gamma$  of all holomorphic maps  $\mathbb{C} \rightarrow GL_r(\mathbb{C})$ . However, even in the case  $r = 1$  this group is larger than we need: in that case we usually work with the group of functions of the form  $\gamma(z) = e^{p(z)}$  where  $p$  is a *polynomial*, and there is no advantage in letting  $p$  be an arbitrary entire function. That would even be confusing. One reason is that if  $p$  is a polynomial, then the action of  $\gamma$  on ideals in the Weyl algebra  $A$  comes from the automorphism  $D \mapsto e^{p(z)} D e^{-p(z)}$  of  $A$ ; if  $p$  is not a polynomial, this map does not preserve  $A$ . For  $r > 1$ , the maps of the form  $\gamma(z) = \exp(p(z))$  (where  $p$  is a polynomial map  $\mathbb{C} \rightarrow \mathfrak{gl}_r(\mathbb{C})$ ) do not form a group. Of course, we can form the subgroup of  $\Gamma$  which they generate, but it is not clear to me what this is. Perhaps for some purposes a smaller group still will suffice.

Note first that the subgroup  $\mathbb{C}I \subset \Gamma_{sc}$  of *constant* scalar matrices acts trivially on the spaces  $\mathcal{C}$  (see Proposition 2.3), so it is really the quotient group that acts; let us change notation so that from now on  $\Gamma$  denotes this quotient group. We then have the decomposition

$$\Gamma = \Gamma_{sc} \times \Gamma_1 ,$$

where  $\Gamma_1$  is the subgroup of maps  $\gamma$  such that  $\det \gamma$  is a scalar constant. Let  $\Gamma_{sc}^{alg}$  be the subgroup of  $\Gamma_{sc}$  consisting of all maps of the form  $\gamma(z) = e^{p(z)} I$  where  $p$  is a *polynomial* (say with zero constant term), and set

$$(8.1) \quad \Gamma^{alg} := \Gamma_{sc}^{alg} \times PGL_r(\mathbb{C}[z]) \subset \Gamma .$$

From an algebraic point of view this seems a quite natural group (see the next section); however, it has the disadvantage that it does not contain the 1-parameter subgroups  $\{\exp(\alpha z^k t)\}$  of  $\Gamma$  unless  $\alpha$  is nilpotent.

As well as  $\Gamma$ , there is the “opposite” group  $\Gamma'$  of symplectomorphisms of each  $\mathcal{C}_n$ , obtained by reversing the roles of  $(X; v)$  and  $(Y; w)$ . We have  $\Gamma' = b\Gamma b$  where  $b$  is the bispectral involution (7.2) (note that  $b$  is *anti*-symplectic). We might ask whether  $\Gamma$  and  $\Gamma'$  generate the symplectomorphism group, but this kind of question seems too difficult. More accessible should be the

**Conjecture 8.1.** The group generated by  $\Gamma^{alg}$  and  $(\Gamma')^{alg} := b\Gamma^{alg}b$  acts transitively on  $\mathcal{C}_n$ .

For  $r = 1$  this is proved in [BW2]. For  $r = 2$  it is presumably the same as the result proved in [BP]; that is, I guess that our group is the same as the group of tame automorphisms of the quiver considered in [BP], but that is not clear to me at present.

It is not very clear where “the” group referred to in Conjecture 8.1 is supposed to live: perhaps in the automorphism group of some quiver? or in  $\text{Aut } M_r(A)$  (see the next section)? For  $r > 1$  it is not clear to me that  $\text{Aut } M_r(A)$  acts on our spaces (even the fat submodules).

There are natural inclusions  $\mathcal{C}_n(r) \hookrightarrow \mathcal{C}_n(r+1)$  given by<sup>2</sup> adding a column of zeros to  $v$  and a row of zeros to  $w$ : these inclusions are compatible with the usual embeddings of  $\Gamma(r)$  into  $\Gamma(r+1)$ . So we might hope to prove Conjecture 8.1 by induction on  $r$ , or even by direct reduction to the case  $r = 1$ , as in [BP].

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<sup>2</sup>If we interpret  $\mathcal{C}_n(r)$  as a space of bundles over a noncommutative  $\mathbb{P}^2$ , this corresponds to adding a trivial line bundle. On the level of  $\text{Gr}^{\text{ad}}$  it sends  $W$  to  $W \oplus \mathbb{C}[z]$ .

9.  $\Gamma$  AND AUTOMORPHISMS

Let  $A^{an}$  momentarily denote the algebra of differential operators with entire coefficients. Then for each  $\gamma \in \Gamma$ , the map  $D \mapsto \gamma D \gamma^{-1}$  is an automorphism of  $M_r(A^{an})$ . In general this map does not preserve the subalgebra  $M_r(A)$ . More precisely, we have the following.

**Proposition 9.1.** *The subgroup of  $\Gamma$  which preserves  $M_r(A)$  is exactly  $\Gamma^{alg}$ .*

*Proof.* It is obvious that  $\Gamma^{alg}$  preserves  $M_r(A)$ . Conversely, suppose that  $\gamma$  preserves  $M_r(A)$ . Since  $\gamma(\partial I)\gamma^{-1} = \partial I - \gamma'\gamma^{-1}$ ,  $\gamma$  must satisfy the condition

$$(9.1) \quad \gamma'\gamma^{-1} \text{ is a polynomial matrix.}$$

Also,  $\gamma$  must satisfy the condition

$$(9.2) \quad \gamma e_{rs} \gamma^{-1} \text{ is a polynomial matrix for all } (r, s),$$

where  $e_{rs}$  is the matrix with 1 in the place  $(r, s)$  and zeros elsewhere. To see that these conditions force  $\gamma$  to belong to  $\Gamma^{alg}$ , write it in the form  $\gamma(z) = e^{p(z)}\gamma_1(z)$ , where  $\det \gamma_1 \equiv 1$ . Then  $\gamma'\gamma^{-1} = p'(z)I + \gamma_1'\gamma_1^{-1}$ . Here the second term has trace zero, so (9.1) shows that  $p'$ , hence also  $p$ , is a polynomial. It remains to show that  $\gamma_1$  is polynomial. Clearly,  $\gamma_1$  satisfies the same condition (9.2) as  $\gamma$ . That this implies that it is polynomial will follow from the stronger statement: if  $A \in \Gamma$  and  $B \in \Gamma_1$  and all the matrices  $A(z)e_{rs}B(z)$  are polynomial, then  $A$  is polynomial. Indeed, the  $(i, j)$  entry in  $Ae_{rs}B$  is  $A_{ir}B_{sj}$ , so our condition says that the product of any entry in  $A$  with any entry in  $B$  is a polynomial. Fix an entry in  $A$ , say  $a(z)$ ; then for all  $(i, j)$  we have

$$a(z)B_{ij}(z) = p_{ij}(z)$$

where the  $p_{ij}$  are polynomials. Since  $\det B \equiv 1$ , taking determinants gives  $a(z)^r = \det(p_{ij})$ . Thus  $a(z)^r$  is a polynomial; since  $a$  is entire, this implies that  $a(z)$  is a polynomial, as required.  $\square$

In the next proposition we regard  $\Gamma^{alg}$  as a subgroup of  $\text{Aut } M_r(A)$ .

**Proposition 9.2.**  *$\Gamma^{alg}$  is exactly the isotropy group of  $zI \in M_r(A)$ .*

*Proof.* It is obvious that  $\Gamma^{alg}$  fixes  $zI$ . The converse depends on the fact that  $\text{Aut } M_r(A)$  is the semi-direct product of  $\text{Inn } M_r(A)$  and  $\text{Aut } A$  (acting on each matrix entry); thus every automorphism has the form  $D \mapsto T\sigma(D)T^{-1}$  for some  $\sigma \in \text{Aut } A$ ,  $T \in GL_r(A)$ . If this fixes  $zI$ , then we have  $zI = T\sigma(z)IT^{-1}$ , or  $zT = T\sigma(z)$ . Looking at the leading coefficient of the matrix operator  $T$ , we conclude that  $\sigma(z) = z$ , that is,  $\sigma \in \Gamma_{sc}^{alg}$ . We now have  $zT = Tz$ , which implies that  $T$  is an operator of order zero in  $\partial_z$ , hence belongs to  $GL_r(\mathbb{C}[z])$  as claimed.  $\square$

For completeness, we give the proof that  $\text{Aut } M_r(A)$  is a semi-direct product, as claimed above. For any algebra  $A$  we have a commutative diagram

$$\begin{array}{ccc} \text{Aut } A & \longrightarrow & \text{Pic } A \\ \downarrow \iota & \searrow & \downarrow \simeq \\ \text{Inn } M_r(A) \hookrightarrow \text{Aut } M_r(A) & \longrightarrow & \text{Pic } M_r(A) \end{array}$$

where the inclusion  $\iota$  makes an automorphism of  $A$  act separately on each entry of a matrix. If  $A$  is the Weyl algebra, then the top horizontal arrow, and hence also the dotted diagonal arrow, is an isomorphism. We may use this dotted arrow to identify  $\text{Aut } A$  with  $\text{Out } M_r(A)$ ; the vertical arrow  $\iota$  then splits the exact sequence in the second row of the diagram.

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